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STEADILY MOVING SOURCES ON THE INTERFACE BETWEEN TWO MEDIA

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Introduction

We shall consider the propagation of acoustic disturbances generated be a moving point source, in the presence of a plane refracting surface. We specify that the source moves, with a constant velocity V, parallel to a plane surface separating two media of different densities and sound velocities. In a previous paper, the author (1960) gave a solution which was valid when the velocity V is greater than the sound velocity in both media, for the source moving along the interface; this solution was found by assuming conical motion and then specifying the nature of the source. The method is suitable neither in the case when the source is off the interface, nor when its velocity is subsonic with respect to one of the media.

In the half-space $y > 0$ we find a medium 1 of density ρ_1 and of sound velocity c_1; in $y < 0$ we have a medium 2 with density $(\rho_2 = \kappa\rho_1)$

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and sound velocity c_2. (We take $c_2 < c_1$ without losing generality.) With $i = 1$ or 2 we may define velocity potentials ϕ_i for the appropriate medium, such that the velocity vector $\underline{q}$ and the pressure change P are given by the equations

$$q = \nabla\phi_i \quad , \qquad P = \rho_i \partial\phi_i/\partial t \quad ,$$

while the ϕ_i satisfy the wave equations

$$\nabla^2\phi_i = \frac{1}{c_i^2}\frac{\partial^2\phi_i}{\partial t^2} \quad . \tag{1}$$

The mathematical problem is to find the potentials ϕ_i which while satisfying the appropriate wave equation also represent disturbances set up by a steadily moving source, and which on the interface satisfy the conditions that the pressure and normal velocity be continuous. These conditions are that when $y = 0$

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial\phi_2}{\partial y} \quad , \quad \text{and} \quad \frac{\partial\phi_1}{\partial t} = \kappa\frac{\partial\phi_2}{\partial t} \quad .$$

Imagine the point source to have a constant velocity V in the z-direction. This steady motion implies that the variables t and z are not independent, but must be related by a relation

$$\tau = t - zV^{-1} \quad ;$$

then the potentials must satisfy a reduced wave equation

$$\frac{\partial^2\phi_i}{\partial x^2} + \frac{\partial^2\phi_i}{\partial y^2} = \frac{1}{\gamma_i^2}\frac{\partial^2\phi_i}{\partial\tau^2} \tag{2}$$

where

$$\gamma_i = c_i V(V^2 - c_i^2)^{-\frac{1}{2}} \quad ,$$

while the continuity conditions for $y = 0$ become

$$\frac{\partial\phi_1}{\partial y} = \frac{\partial\phi_2}{\partial y} \quad , \qquad \frac{\partial\phi_1}{\partial\tau} = \kappa\frac{\partial\phi_2}{\partial\tau} \tag{3}$$

It is clear that in the limiting case $V = \infty$, the moving point source is equivalent to an infinite transient line source. It is also clear that although γ_i represents a real transverse velocity when $V > c_i$, it is imaginary when $V < c_i$; the nature of the reduced wave equation 2 changes from hyperbolic to elliptic and the form of the acoustic disturbance will also change from one which permits discontinuities across characteristic surfaces to one which is continuous everywhere. When only one medium is present, the change in the nature of the acoustic disturbance as the source velocity changes from supersonic to subsonic is well known (See e.g., Ward 1955). For the two-medium problem previous work has been restricted to the examination of fixed point- or line- sources of a given time dependence. As well as mentioning the classical work of Sommerfeld (e.g. 1949) we may cite the work of Cagniard

(1939) and Pekeris (1956) for accounts of the way in which integral transforms may be used. For the fixed point source of arbitrary time dependence there is an axis of symmetry, for the uniform infinite line source of arbitrary time dependence there is a plane of symmetry, but when the point source is moving with a finite velocity, there is no such property, and the calculations in the two-medium problem using integral transforms are not easy. The method described below is suggested as a useful alternative.

Section 2: The point source moving in a single uniform medium

The concept of a point source of unit strength moving in a single medium is a well established one (See e.g. Ward 1955). When such a source moves at a steady supersonic velocity V, the associated velocity potential, which satisfies the three-dimensional wave equation with sound velocity c, is

$$\phi = \frac{-\gamma}{2\pi(\gamma^2\tau^2 - r^2)^{\frac{1}{2}}} \qquad (4)$$

within the conical region $\gamma\tau > (x^2 + y^2)^{\frac{1}{2}} = r > 0$,

where

$$\gamma = cV(V^2 - c^2)^{-\frac{1}{2}}$$

and

$$\tau = t - zV^{-1} \quad .$$

When the source moves at a steady subsonic velocity V the potential is

$$\phi = \frac{-\mu}{4\pi(\mu^2\tau^2 + r^2)^{\frac{1}{2}}} \qquad (5)$$

where

$$\mu = cV(c^2 - V^2)^{-\frac{1}{2}}$$

An alternative form which combines both these results, is that for any fixed value V, the point source of unit strength has a velocity potential given in the region $x > 0$, $y > 0$ by the formula

$$\phi = -\frac{1}{4\pi^2} \, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma \, dp}{(1-p^2)^{\frac{1}{2}} \, [\gamma\tau - px - y(1-p^2)^{\frac{1}{2}}]} \tag{6}$$

When $V < c$ we have a convention that $\gamma < i\mu$. We also take the integration path along the real axis of the complex p-plane to pass above the branch point at $p = 1$ and below that at $p = -1$. For points in other quadrants of the transverse (x, y) plane the signs in the factor

$$[\gamma\tau \pm px \pm y(1-p^2)^{\frac{1}{2}}]^{-1}$$

are chosen and used in the integral so as to make the limiting two-dimensional solution with $V = \infty$, $\gamma = c$ and $\tau = t$ represent a disturbance which travels outwards from the z-axis.

The integral 6 which is a superposition of parametric plane solutions of the reduced wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{\gamma^2} \frac{\partial^2 \phi}{\partial \tau^2}$$

obviously represents a disturbance which travels in the z-direction with a velocity V. To show its equivalence to the known potentials 4 and 5, we consider separately the cases $V > c$ and $V < c$.

We change to polar coordinates in the transverse plane, so that we must consider the integral

$$\phi = -\frac{1}{4\pi^2} \, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma \, dp}{(1-p^2)^{\frac{1}{2}}\{\gamma\tau - r[p\cos\theta + \sin\theta(1-p^2)^{\frac{1}{2}}]\}} \tag{7}$$

in the range $0 < \theta < \pi/2$.

Apart from the branch points at $p = \pm 1$, the only singularities of this integrand are simple poles at the points where the function

$$\{\gamma\tau - r[p\cos\theta + \sin\theta[1-p^2]^{\frac{1}{2}}]\}$$

vanishes. When, for $V > c$, γ is a real velocity, this function vanishes only when $\tau > 0$, and then at the points

$$p = [\gamma\tau\cos\theta \pm i\sin\theta(\gamma^2\tau^2 - r^2)^{\frac{1}{2}}]/r \qquad \text{for } \gamma\tau > r > 0$$

and

$$p = [\gamma\tau\cos\theta \pm \sin\theta(r^2 - \gamma^2\tau^2)^{\frac{1}{2}}]/r \qquad \text{for } r > \gamma\tau > 0$$

For a fixed value of θ the locus of the complex poles for all positive values of the ratio r/τ is the branch of a hyperbola given in the right-hand half p-plane by the equation $p = \cosh(v + i\theta)$ for real values of v. When $r \neq 0$, the integrand is $0(p^{-2})$ as $|p| \to \infty$; the integration path can therefore be shifted by a rotation in the positive sense into the hyperbolic path mentioned above and shown in Figure 1 without change of value. Small deformations must be made in this path to avoid the two conjugate poles in the sense shown (for a given pair of values r and τ), and the single real pole to the left of the hyperbola must also be noted. Only the residues at the two poles on the hyperbola contribute to the integral 7, and the formula 4 is easily recovered. The residue at the pole on the real axis is real and makes no contribution to the integral, while the principal value of the integral along the hyperbola is easily seen to be imaginary and hence of no account. (By considering separately the two halves of the hyperbolic path we arrive at an integral which is the difference of complex conjugates.) In the absence of poles when $\tau < 0$, the integration path is shifted into a horizontal loop between the points $p = 1$ and $p = \infty$; there is no contribution to ϕ in this case.

When γ $(= i\mu)$ is imaginary the simple poles of the integrand are for a given θ at the points

$$p = [\, i\mu\tau \cos\theta \pm \sin\theta (r^2 + \mu^2\tau^2)^{\frac{1}{2}}] / r \quad ,$$

these points being either in the upper half p-plane when $\tau > 0$ or in the lower half p-plane when $\tau < 0$. The pole in the right-hand half plane lies on the hyperbola $p = i \sinh(v - i\theta)$ for real values of v, and it is to this hyperbola that we shift the integration path. Two distinct situations arise. When $\tau > 0$ only the pole in the first quadrant need be considered in the integration. When $\tau < 0$, both the pole in the third quadrant and that in the fourth quadrant make a contribution. These points and the sense in which the poles are encircled are shown in Figures 2a and 2b. Again the principal value of the integral along the hyperbola is imaginary leaving only the residues at the poles to fix the potential, and, regardless of the sign of τ the result is given by equation (5).

To calculate the strength of the point source from the potential 7 we shall use an indirect method. A direct method involves the calculation of volume flux across a closed surface which contains the source. With the usual aim of reducing the subsequent integration to one over the range of a single variable this surface is chosen differently according to the specific value of V. When $V < c$ an ellipsoid is taken, and when $V > c$, a hyperboloid. The indirect approach avoids the need for individual treatment.

Starting from the definition 7 for ϕ in the region $r > 0$, $0 < \theta < \pi/2$ we make several statements. These are

1) The quantity $\tau\phi$ is a function only of the variables $s\ (= r/\tau)$ and θ.

2) The quantity Q which satisfies the same transverse wave equation as ϕ and also the relation $\phi = \partial Q/\partial\tau$ is given by the equation

$$Q = -\frac{1}{4\pi^2}\ \mathrm{Rl}\int_{-\infty}^{\infty}\frac{\ln\{\gamma\tau - r[p\cos\theta + \sin\theta(1-p^2)^{\frac{1}{2}}\}\,dp}{(1-p^2)^{\frac{1}{2}}}$$

It follows that

$$\tau\phi = \tau\partial Q/\partial\tau = -\ s\partial Q/\partial s = -r\partial Q/\partial r\ .$$

3) Q is an integral superposition of logarithmic wave potentials, and is therefore the velocity potential for some distribution of point-sources along the z-axis.

4) The strength when $\tau \neq 0$ of such a line distribution is found by calculating the volume flux per unit length S_Q across a cylindrical surface which contains the z-axis. Thus

$$S_Q = 4 \int_0^{\pi/2} \lim_{r\to 0} (r\partial Q/\partial r)\, d\theta = -4 \int_0^{\pi/2} \lim_{r\to 0} (\tau\phi)\, d\theta \ . \tag{8}$$

5) Because of the relation between ϕ and Q the quantity $S_\phi = \partial S_Q/\partial\tau$ is the volume flux <u>per unit length</u> created by the source whose potential is ϕ. The total volume flux created by this point source, found by integrating S_ϕ with respect to τ must be equal to the discontinuity $S_Q(\tau = 0_+) - S_Q(\tau = 0_-)$.

6) It is simple to find the quantity $\lim_{r\to 0}(\tau\phi)$ from the equations (4) and (5). From (4) and (8) it follows that $S_Q = U(\tau)$ when $V > c$ $[U(\tau) = 1$ when $\tau > 0$, $U(\tau) = 0$ when $\tau < 0]$, while from (5) and (8) $S_Q = \mathrm{sgn}(\tau)/2$ $[\mathrm{sgn}(\tau) = 1$ if $\tau > 0$, $\mathrm{sgn}(\tau) = -1$ if $\tau < 0]$ when $V < c$. Thus for all values of V the potential 7 is associated with a source of <u>unit</u> strength. The distinction between the subsonic and supersonic interpretations of the potential Q is rather curious. When $V > c$, Q is the potential of a semi-infinite line source moving lengthways with velocity V. When $V < c$, Q is the potential of an <u>infinite</u> line singularity

moving lengthways with velocity V, the singularity behind the point $z = Vt$ being a uniform line source, and that ahead of this point being a uniform line sink.

The potential of a lengthways moving uniform line source of finite length may easily be found by simple superposition. Even though the fundamental line potentials Q have different interpretations according to the value of V, there is no such difference in the case of finite length. Thus, for a length L of a line source, the strength is defined either by the difference $U(\tau) - U(\tau + LV^{-1})$ or the difference $\frac{1}{2}\,\mathrm{sgn}(\tau) - \frac{1}{2}\,\mathrm{sgn}(\tau + LV^{-1})$; regardless of the value of V the potential Q_L must be given by the equation

$$Q_L = -\frac{1}{4\pi^2}\ \mathrm{Rl}\int_{-\infty}^{\infty}(1-p^2)^{-\frac{1}{2}}\ln\left\{\frac{\gamma\tau - r[p\cos\theta + \sin\theta(1-p^2)^{\frac{1}{2}}}{\gamma(\tau + LV^{-1}) - r[p\cos\theta + \sin\theta(1-p^2)^{\frac{1}{2}}}\right\}dp \tag{9}$$

Section 3: A point source moving parallel to a plane interface

In this section we return to the discussion of the two-medium problem. We take a point source of unit strength moving in medium 2 with steady velocity V at a constant distance y_2 from the interface. We have expressed the potential of this source as a superposition of parametric plane waves and we are able to find the complete potential by considering the reflection and refraction of these plane waves and then writing down the integral superposition to give the total field.

We restrict attention to the region $x > 0$. We have three distinct regions of interest in this half of the transverse plane. For $0 > y > -y_2$, the source field travels in the positive y-direction and is reflected in the plane $y = 0$; this reflected field will apparently be produced at an image source at the point $x = 0$, $y = -y_2$ and will travel in the negative y-direction. Thus we may write

$$\phi_2 = -\frac{1}{4\pi^2}\, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2 dp}{(1-p^2)^{\frac{1}{2}}} \left\{ \frac{1}{\gamma_2\tau - px - (y+y_2)(1-p^2)^{\frac{1}{2}}} + \frac{R\,(p)}{\gamma_2\tau - px + (y-y_2)(1-p^2)^{\frac{1}{2}}} \right\} \tag{10}$$

when $x > 0$, $0 > y > -y_2$,

$R(p)$ being the reflection coefficient of the plane wave characterized by the parameter p. On the surface $y = 0$ both the primary and image fields are superpositions of plane waves moving along the interface with velocity $\gamma_2 p^{-1}$ and in which the variables x and τ appear in the function $[\gamma_2\tau - px - y_2(1-p^2)^{\frac{1}{2}}]^{-1}$.

The parametric plane wave solution of the reduced wave equation

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = \frac{1}{\gamma_1^2}\,\frac{\partial^2\phi_1}{\partial \tau^2}$$

in medium 1 which has the same form and velocity at the interface and which travels in the positive y-direction must have the form

$$[\gamma_2\tau - px - y_2(1-p^2)^{\frac{1}{2}} - y(m^2 - p^2)^{\frac{1}{2}}]^{-1}$$

where $m = \gamma_2/\gamma_1$. Hence the potential in medium 1 is given by

$$\phi_1 = -\frac{1}{4\pi^2}\,\mathrm{Rl}\int_{-\infty}^{\infty}\frac{\gamma_2\;T(p)\,dp}{(1-p^2)^{\frac{1}{2}}\{\gamma_2\tau - px - y_2(1-p^2)^{\frac{1}{2}} - y(m^2-p^2)^{\frac{1}{2}}\}} \tag{11}$$

when $x > 0$, $y > 0$, and where $T(p)$ is the transmission coefficient for the parametric plane wave.

For the potentials 10 and 11 to satisfy the continuity conditions 3 at the interface, the elementary plane waves must also do so. Hence we derive the conditions

$$\kappa[1 + R(p)] = T(p)$$

and

$$(1-p^2)^{\frac{1}{2}}[1 - R(p)] = (m^2 - p^2)^{\frac{1}{2}}\,T(p)$$

whence

$$R(p) = \frac{(1-p^2)^{\frac{1}{2}} - \kappa(m^2-p^2)^{\frac{1}{2}}}{(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}}$$

and

$$T(p) = \frac{2\kappa(1-p^2)^{\frac{1}{2}}}{(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}} \tag{12}$$

With these results only the field in the region $x > 0$, $-y_2 > y$ remains to be given. This is naturally written down as the sum of the primary source field and of the image field, each being a superposition of plane waves which travel in the negative y-direction.

Thus for $x > 0$, $y > 0$,

$$\phi_1 = -\frac{\kappa}{2\pi^2} \quad \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2 \, dp}{[(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}][\gamma_2\tau - px - y_2(1-p^2)^{\frac{1}{2}} - y(m^2-p^2)^{\frac{1}{2}}]} \tag{13}$$

for $x > 0$, $0 > y > -y_2$

$$\phi_2 = -\frac{1}{4\pi^2} \quad \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2(1-p^2)^{-\frac{1}{2}} dp}{[(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}]} \left\{ \frac{(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}}{\gamma_2\tau - px - (y+y_2)(1-p^2)^{\frac{1}{2}}} + \frac{(1-p^2)^{\frac{1}{2}} - \kappa(m^2-p^2)^{\frac{1}{2}}}{\gamma_2\tau - px + (y-y_2)(1-p^2)^{\frac{1}{2}}} \right\} \tag{14}$$

and for $x > 0$, $y < -y_2$

$$\phi_2 = -\frac{1}{4\pi^2} \quad \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2(1-p^2)^{-\frac{1}{2}} dp}{[(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}]} \left\{ \frac{(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}}{\gamma_2\tau - px + (y+y_2)(1-p^2)^{\frac{1}{2}}} + \frac{(1-p^2)^{\frac{1}{2}} - \kappa(m^2-p^2)^{\frac{1}{2}}}{\gamma_2\tau - px + (y-y_2)(1-p^2)^{\frac{1}{2}}} \right\} \tag{15}$$

The imputation of a direction of motion for the elementary waves is strictly true only when the quantities γ_2 and m are real. The method of this paper rests, however, on the basis that a solution which is correct for one range of values of V is also correct for other ranges, since neither the three dimensional wave equation nor the continuity conditions at the interface depend on the value of V. The solutions to be displayed in later sections of this paper may all, moreover, be verified a posteriori .

When the source is taken in medium 1 to move at a constant height y_1 above the interface, we find in a similar manner the potentials

$$\phi_2 = -\frac{1}{2\pi^2}\, \mathrm{Rl}\int_{-\infty}^{\infty} \frac{\gamma_2\, dp}{[(1-p^2)^{\frac{1}{2}}+\kappa(m^2-p^2)^{\frac{1}{2}}][\gamma_2\tau-px-y_1(m^2-p^2)^{\frac{1}{2}}+y(1-p^2)^{\frac{1}{2}}]} \tag{16}$$

for $x > 0$, $y < 0$,

$$\phi_1 = -\frac{1}{4\pi^2}\, \mathrm{Rl}\int_{-\infty}^{\infty} \frac{\gamma_2(m^2-p^2)^{-\frac{1}{2}} dp}{[(1-p^2)^{\frac{1}{2}}+\kappa(m^2-p^2)^{\frac{1}{2}}]} \left\{ \frac{\kappa(m^2-p^2)^{\frac{1}{2}}+(1-p^2)^{\frac{1}{2}}}{\gamma_2\tau-px+(y-y_1)(m^2-p^2)^{\frac{1}{2}}} + \frac{\kappa(m^2-p^2)^{\frac{1}{2}}-(1-p^2)^{\frac{1}{2}}}{\gamma_2\tau-px-(y+y_1)(m^2-p^2)^{\frac{1}{2}}} \right\} \tag{17}$$

for $x > 0$, $0 < y < y_1$, and

$$\phi_1 = -\frac{1}{4\pi^2}\, \mathrm{Rl}\int_{-\infty}^{\infty} \frac{\gamma_2(m^2-p^2)^{-\frac{1}{2}} dp}{[(1-p^2)^{\frac{1}{2}}+\kappa(m^2-p^2)^{\frac{1}{2}}]} \left\{ \frac{\kappa(m^2-p^2)^{\frac{1}{2}}+(1-p^2)^{\frac{1}{2}}}{\gamma_2\tau-px-(y-y_1)(m^2-p^2)^{\frac{1}{2}}} + \frac{\kappa(m^2-p^2)^{\frac{1}{2}}-(1-p^2)^{\frac{1}{2}}}{\gamma_2\tau-px-(y+y_1)(m^2-p^2)^{\frac{1}{2}}} \right\} \tag{18}$$

for $x > 0$, $y > y_1$.

The six formulae from equation (13) to (18) are taken to define the potential field for any fixed (real) value of V. We shall take these results no further in this paper; special attention will, however, be given to the limiting case with the point source moving on the interface. To do this we can allow the distances y_1 or y_2 to approach zero. However we carry out this process, we have the expressions

$$\phi_2 = -\frac{1}{2\pi^2} \, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2 \, dp}{[(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}][\gamma_2\tau - px + y(1-p^2)^{\frac{1}{2}}]} \tag{19}$$

for $x > 0$, $y < 0$, and

$$\phi_1 = -\frac{\kappa}{2\pi^2} \, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2 \, dp}{[(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}][\gamma_2\tau - px - y(m^2-p^2)^{\frac{1}{2}}]} \tag{20}$$

for $x > 0$, $y > 0$.

At this point it is worth our while to consider the result of letting two sources of different strengths, one in each medium, coalesce into a single source moving on the interface. If the sum of the individual strengths is a constant T, then the potentials for the limiting point source are easily shown to be multiples of the potentials 19 and 20 by the factor T. The implication is that the form of the point source potential is as unique on the interface as it is in the interior of a uniform medium.

We may add here the statement that with $i = 1$ or 2 the quantity $\lim_{v \to 0} [\phi_i/V]$ satisfies Laplace's equation, so that we can pick out from the above equation not only potentials associated with the moving point source, but also the steady potentials associated with the fixed point source as well. The problem of the fixed transient point source will be solved elsewhere.

In the remainder of this paper we shall examine the formulae (19) and (20) in greater detail.

Section 4: The evaluation of velocity potential in the medium with smaller sound velocity

The general formula, given in terms of polar coordinates r and θ,

$$\phi_2 = -\frac{1}{2\pi^2} \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2 \, dp}{[(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}\{\gamma_2\tau - r[p\cos\theta - (1-p^2)^{-\frac{1}{2}}\sin\theta]\}} \qquad (21)$$

represents the velocity potential in medium 2 when $0 > \theta > \pi/2$, but the process of simplifying this integral depends on which of three possible ranges of value the longitudinal velocity has, since this is bound to affect the position of the singularities of the integrand.

When $V > c_1 > c_2$, the transverse velocities γ_2 and γ_1 are real with $m (= \gamma_2/\gamma_1) < 1$. The integrand of (21) has branch points at $p = \pm 1$, $p = \pm m$ on the real axis, and when $\tau < 0$ no other singularities. When $\tau > 0$ it has simple poles at the points

$$p = [\gamma_2\tau\cos\theta \pm i\sin\theta(\gamma_2^2\tau^2 - r^2)^{\frac{1}{2}}]/r \qquad \text{when} \quad \gamma_2\tau > r > 0$$

or

$$p = [\gamma_2\tau\cos\theta \pm \sin\theta(r^2 - \gamma_2^2\tau^2)^{\frac{1}{2}}]/r \qquad \text{when} \quad r > \gamma_2\tau > 0 .$$

For a given value of θ the locus of all the possible complex poles is the hyperbola $p = \cosh(v - i\theta)$ where v is real. With the formal horizontal integration path taken to pass below the branch points $p = -1$ and $p = -m$, and above the points $p = +1$ and $p = +m$, we take a new integration path along the hyperbola, apart from deformations to avoid the two simple conjugate poles which correspond to particular values of r, τ and θ. Then if we also include the (single) simple pole on the left of the hyperbola in the calculation, there will be no change in the value of the integral when $r \neq 0$ since the integrand is $0(p^{-2})$ as $|p| \to \infty$. A distinction is necessary, resulting in two different integration paths as shown in Figures 3a and 3b. The hyperbola in each case crosses the real p-axis at the point $p = \cos\theta$. When $\cos\theta > m$ the hyperbola passes to the right of the branch point $p = +m$, so that as shown in Figure 3b a horizontal loop integral between the points $p = m$ and $p = \cos\theta$ must be included in the new integration path. When, as in Figure 3a, $\cos\theta < m$, no such loop is needed.

The pole at the real point $p = [\gamma_2\tau\cos\theta + \sin\theta(r^2 - \gamma_2^2\tau^2)^{\frac{1}{2}}]/r = \cos\alpha$, say, is present in both these cases, but when the residue at this point is real there is no contribution to the potential. Thus only in the case $\cos\theta > m$ and then when $\cos\alpha > m$ can there be a contribution to the potential, because the residue now has an imaginary part.

When $\gamma_2\tau > r > 0$ the contribution from the two poles on the hyperbola is

$$\phi_2 = -\frac{\gamma_2}{\pi(\gamma_2^2\tau^2 - r^2)^{\frac{1}{2}}} \, \mathrm{Rl}\left[\frac{\sinh(v - i\theta)}{\sinh(v - i\theta) + \kappa[\cosh^2(v - i\theta) - m^2]^{\frac{1}{2}}}\right] \quad v = \operatorname{arc\,cosh} \gamma_2\tau/r \tag{22}$$

for the full range $0 > \theta > -\pi/2$, that is for both Figure 3a and 3b. When the residue at the real pole also is included we have the further contribution

$$\phi_2 = + \frac{\kappa\gamma_2}{\pi(r^2-\gamma_2^2\tau^2)^{\frac{1}{2}}} \quad \frac{\sin\alpha[\cos^2\alpha - m^2]^{\frac{1}{2}}}{\sin^2\alpha + \kappa^2(\cos^2\alpha - m^2)} \tag{23}$$

in a region for $r > \gamma_2\tau > 0$ restricted to values of $\alpha = -\theta + \arccos \gamma_2\tau/r$ which lie in the range $1 > \cos\alpha > m$. Equation (22) defines the potential in a sector $0 > \theta > -\pi/2$ of the interior of a cone $r = \gamma_2\tau$. Equation (23) defines the potential in a 'head-wave' region contained between the cone $r = \gamma_2\tau$, the tangent plane which meets the interface $\theta = 0$ at the line $x = \gamma_1\tau$ for $\tau > 0$, and the plane $y = 0$.

When for negative values of τ there are no poles of the integrand, the horizontal integration path can be replaced by a loop integral round the branch points $p = 1$ and $p = m$; this integral is imaginary and makes no contribution.

When $V < c_2 < c_1$, the transverse velocities γ_2 and γ_1 are positive imaginary, leaving the ratio m real in the range $m > 1$. In this fully subsonic case the explicit form of 21 to be evaluated is

$$\phi_2 = -\frac{1}{2\pi^2} \text{ Rl} \int_{-\infty}^{\infty} \frac{i\mu_2\, dp}{[(1-p^2)^{\frac{1}{2}}+\kappa(m^2-p^2)^{\frac{1}{2}}]\{i\mu_2\tau - r[p\cos\theta - (1-p^2)^{\frac{1}{2}}\sin\theta]\}} \tag{24}$$

with the formal integration path along the real axis indented so as to pass above the points $p = 1$, $p = m$ and below the points $p = -1$ and $p = -m$. The integrand 24 has simple poles at the points

$$p = [i\mu_2\tau \cos\theta \pm \sin\theta(r^2 + \mu_2^2\tau^2)^{\frac{1}{2}}]/r \qquad ;$$

for a given θ the locus of poles in the right-hand half-plane is the hyperbola $p = i\sinh(v + i\theta)$ for real values of v. When $\tau > 0$ for specified r and θ we may shift the integration path into this hyperbola without change of value, with only a pole in the first quadrant to be taken into account (See Figure 4a). When $\tau < 0$ the same shift of path is possible, but now poles both in the third and the fourth quadrants must be considered (See Figure 4b). The formula for the potential turns out to be independent of the sign of τ. We have the result that

$$\phi_2 = -\frac{\mu_2}{2\pi[r^2+\mu_2^2\tau^2]^{\frac{1}{2}}} \; \mathrm{Rl}\left[\frac{\cosh(v+i\theta)}{\cosh(v+i\theta)+\kappa[m^2+\sinh^2(v+i\theta)]^{\frac{1}{2}}}\right]_{v = \text{arc}\sinh \mu_2\tau/r}$$
$$+\frac{\mu_2 r}{\pi^2}\,\mathrm{Im}\int_0^\infty \frac{\sinh v}{(\mu_2^2\tau^2 - r^2\sinh^2 v)}\;\frac{\cosh(v+i\theta)\,dV}{[\cosh(v+i\theta)+\kappa(m^2+\sinh^2(v+i\theta))^{\frac{1}{2}}]} \;; \tag{25}$$

the first term comes from the residue terms, and the second is the principal value of the integral along the hyperbola. Both terms are present without restriction in the whole of the quarter-space $r > 0$, $0 > \theta > -\pi/2$.

In the intermediate case with $c_1 > V > c_2$, γ_2 is real, γ_1 ($= i\mu_1$) is imaginary and the ratio m ($= -in$) is also imaginary. The branch point at $p = m$ thus migrates with change of V to the negative part of the imaginary axis, and the possible integration contour is also changed. We are now examining the integral

$$\phi_2 = -\frac{1}{2\pi^2}\,\mathrm{Rl}\int_{-\infty}^{\infty}\frac{\gamma_2\,dp}{[(1-p^2)^{\frac{1}{2}}-i\kappa(p^2+n^2)^{\frac{1}{2}}]\{\gamma_2\tau-r[p\cos\theta-(1-p^2)^{\frac{1}{2}}\sin\theta]\}} \tag{26}$$

The position of the poles is exactly as in the fully supersonic case, but in shifting the integration path to the hyperbola $p = \cosh(v - i\theta)$ we must, in addition to considering the poles, also include a loop between the points $p = -in$ and $p = -i\infty$ in order to preserve the value of the potential. The full integration path when $\tau > 0$ is shown in Figure 5a while the corresponding integration path when $\tau < 0$ when there are no poles is shown in Figure 5b.

When $\tau > 0$ there are three distinct contributions. From the complex poles we have the residue term

$$\phi_2 = -\frac{\gamma_2}{\pi[\gamma_2^2\tau^2-r^2]^{\frac{1}{2}}}\,\mathrm{Rl}\left[\frac{\sinh(v-i\theta)}{\sinh(v-i\theta)+\kappa[\cosh^2(v-i\theta)+n^2]^{\frac{1}{2}}}\right]_{v=\operatorname{arc\,cosh}\gamma_2\tau/r} \tag{27a}$$

this being the exact continuation of equation (22) within the cone $\gamma_2\tau > r > 0$. The principal part of the integral reduces to an imaginary quantity, so that we have next the residue at $p = \cos\alpha$, this being the head wave contribution

$$\frac{\kappa\gamma_2\sin\alpha\,[\cos^2\alpha+n^2]^{\frac{1}{2}}}{\pi[r^2-\gamma_2^2\tau^2]^{\frac{1}{2}}[\sin^2\alpha+\kappa^2(\cos^2\alpha-m^2)]} \tag{27b}$$

with $\alpha = -\theta + \arccos \gamma_2\tau/r$ in the range $0 < \alpha < \pi/2$, $r > \gamma_2\tau < 0$. This head wave disturbance occupies the whole region $\tau > 0$ outside the cone $r = \gamma_2\tau$.

The contribution from the vertical loop is present for all positive and negative values of τ and has the form

$$\frac{\kappa r \gamma_2 \cos\theta}{\pi^2} \int_n^\infty \frac{dq(q^2 - n^2)^{\frac{1}{2}}}{[1+q^2+\kappa^2(n^2-q^2)]\{(\gamma_2\tau + r\sin\theta(1+q^2)^{\frac{1}{2}})^2 + r^2q^2\cos^2\theta\}} \tag{27c}$$

The horizontal loop contribution which must be considered when $\tau < 0$ in addition that given by equation (27c) has the form

$$\frac{\kappa \gamma_2 r \sin\theta}{\pi^2} \int_1^\infty \frac{(p^2-1)^{\frac{1}{2}}(p^2+n^2)^{\frac{1}{2}}\, dp}{[\kappa^2(p^2+n^2)+1-p^2][(\gamma_2\tau - rp\cos\theta)^2 + (p^2-1)\, r^2\sin^2\theta]} \tag{27d}$$

The general expression for ϕ_2 has now been simplified in three distinct ranges of V. A feature which is held in common for all three ranges is that only the explicit residue term is present on the axis $r = 0$ away from the source; the remaining integral terms are all seen to vanish on this axis away from the source.

Section 5: The evaluation of velocity potential in the medium with the larger sound velocity

The formula to be simplified for various ranges of the longitudinal velocity V is, for $0 < \theta < \pi/2$

$$\phi_1 = -\frac{\kappa}{2\pi^2} \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2 \, dp}{[(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}]\{\gamma_2\tau - r[p\cos\theta + (m^2-p^2)^{\frac{1}{2}}\sin\theta]\}} \tag{28}$$

When $V > c_1 > c_2$ the parameters in the integrand are real; the poles of the integrand at the points

$$p = \left[\gamma_2\tau\cos\theta \pm i\sin\theta\,[\gamma_2^2\tau^2 - m^2r^2]^{\frac{1}{2}}\right]/r, \quad \text{for} \quad \gamma_1\tau > r > 0,$$

lie in the right-hand half of the p-plane on the hyperbola $p = m\cosh(v + i\theta)$ and for real values of v. This hyperbola crosses the real axis at the point $p = m\cos\theta$ which is always to the left of the point $p = m < 1$. We shift the integration path to this hyperbola with indentations at the poles as shown in Figure 6. We note the presence of a pole on the left of the hyperbola at the point

$$p = [\gamma_2\tau\cos\theta - \sin\theta(m^2r^2 - \gamma_2^2\tau^2)^{\frac{1}{2}}]/r$$

for $r > \gamma_1\tau > 0$.

The residue here is real and there is no contribution to the potential. The principal part of the integral along the hyperbola makes no contribution either, since the integral in the first quadrant combines with that in the fourth quadrant to give an imaginary quantity. The residue at the conjugate poles gives the result that

$$\phi_1 = -\frac{\kappa\gamma_2}{\pi[\gamma_1^2\tau^2 - r^2]^{\frac{1}{2}}} \, \mathrm{Rl} \left[\frac{\sinh v + i\theta}{[m^2\cosh^2(v+i\theta)-1]^{\frac{1}{2}} + m\kappa\sinh(v+\theta)} \right] \qquad v = \operatorname{arc\,cosh} \gamma_1\tau/r \tag{29}$$

for $\gamma_1\tau > r > 0$ only.

For $\tau < 0$, in the absence of poles of the integrand, the integration path may be shifted into a loop about the branch points $p = 1$ and $p = m$; this integral may be seen to be imaginary, thus making no contribution to the potential.

We note here that the equations (22), (23) and (29) determine in full the potentials for a supersonic point source moving along the interface; they are equal to these previously obtained by another method (Papadopoulos 1960).

In the fully subsonic case with $V < c_2 < c_1$, and $m = \mu_2/\mu_1 > 1$ we have the integral

$$\phi_1 = -\frac{\kappa}{2\pi^2} \, \mathrm{Rl} \int_{-\infty}^{\infty} \frac{i\mu_2 \, dp}{[(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}]\{i\mu_2\tau - r[p\cos\theta + \sin\theta(m^2-p^2)^{\frac{1}{2}}]\}} \tag{30}$$

Here the poles of the integrand, at the points

$$p = \left[i\mu_2 \tau \cos\theta \pm \sin\theta \left(\frac{r^2}{m^2} + \mu_2^2 \tau^2\right)^{\frac{1}{2}} \right] / r \quad \ldots$$

lie on the hyperbola $p = im \sinh(v - i\theta)$ for real values of v, when they are in the right-hand half p-plane. As before there is a distinction between the number of poles to be taken into consideration in the cases $\tau > 0$ and $\tau < 0$; a further distinction arises because when $m \sin\theta > 1$, the hyperbola $p = im \sinh(v - i\theta)$ passes to the right of the branch point $p = 1$. Thus when $m \sin\theta < 1$, we shift the integration path to the hyperbola, but when $m \sin\theta > 1$, the same shift involves the addition of a horizontal loop integral between the points $p = 1$ and $p = m \sin\theta$. The four cases are indicated in Figure 7.

It is found, however, that the sign of τ has no effect on the formula for the potential. Thus, regardless of this sign and of the value of θ in the range $0 < \theta < \pi/2$, we have the potential

$$\phi_1 = - \frac{\kappa\mu_2}{2\pi[\mu_1^2\tau^2 + r^2]^{\frac{1}{2}}} \, \mathrm{Rl} \left[\frac{\cosh(v - i\theta)}{m\kappa\cosh(v - i\theta) + [m^2 \sinh(v - i\theta) + 1]^{\frac{1}{2}}} \right]_{v = \operatorname{arc\,sinh} \mu_1 \tau / r} \tag{31a}$$

$$+ \frac{\kappa\mu_2 r}{\pi^2} \, \mathrm{Im} \int_0^\infty \frac{\sinh v}{[\mu_1^2\tau^2 - r^2 \sinh^2 v]} \; \frac{\cosh(v - i\theta)}{\{m\kappa\cosh(v - i\theta) + [m^2 \sinh^2(v - i\theta) + 1]^{\frac{1}{2}}\}} \, dV$$

with the first term coming from the residue at the poles, and the second term the principal part of the integral along the hyperbola. In addition, when $m \sin\theta > 1$, we have the contribution from the horizontal loop integral, which reduces to the term

$$-\frac{\mu_2 \kappa r}{\pi^2} \int_1^{m \sin\theta} \frac{(p^2-1)^{\frac{1}{2}}[p\cos\theta + \sin\theta(m^2-p^2)^{\frac{1}{2}}]\, dp}{[p^2-1+\kappa^2(m^2-p^2)]\{\mu^2\tau^2 + r^2[p\cos\theta + (m^2-p^2)^{\frac{1}{2}}\sin\theta]\}} \qquad (31b)$$

The function of this term is to remove a discontinuity in the quantity $\partial\phi_1/\partial\theta$ which appears on the plane $\tau = 0$ in the first term of 31a for values of θ which make $1 > \sin\theta > 1/m$.

Finally in the intermediate case $c_1 > V > c_2$ we examine the formula

$$\phi_1 = -\frac{\kappa}{2\pi^2} \mathrm{Rl} \int_{-\infty}^{\infty} \frac{\gamma_2\, dp}{[(1-p^2)^{\frac{1}{2}} - i\kappa(p^2 \ \ n^2)^{\frac{1}{2}}]\{\gamma_2\tau - r[p\cos\theta - i\sin\theta(p^2+n^2)^{\frac{1}{2}}]\}} \qquad (32)$$

This case differs from the earlier ones in that the poles of the integrand lie at points

$$p = \{\gamma_2\tau\cos\theta \pm i\sin\theta[r^2n^2 + \gamma_2^2\tau^2]^{\frac{1}{2}}\}/r \quad ;$$

for a given θ and real values of v the hyperbola $p = n\sinh(v - i\theta)$ is the locus of these points, in the lower half p-plane. The deformed integration path which includes this hyperbola is shown in Figure 8. Whatever the sign of τ there is only one pole which makes a contribution to the potential. There are two other terms present, the principal part of the integral along the hyperbola, and the horizontal loop integral. Thus the potential ϕ_1 is given in full by the expression

$$\phi_1 = -\frac{\kappa\gamma_2}{2\pi[\mu_1^2\tau^2+r^2]^{\frac{1}{2}}}\,\mathrm{Rl}\left[\frac{\cosh v-i\theta}{[1-n^2\sinh^2(v-i\theta)]^{\frac{1}{2}}-i\kappa n\cosh v-i\theta}\right]_{v=\operatorname{arc\,sinh}\mu_1\tau/r}$$

$$+\frac{\kappa\gamma_2 r}{\pi^2}\,\mathrm{Im}\int_0^\infty\frac{\sinh v\cosh(v-i\theta)\,dV}{(\mu_1^2\tau^2-r^2\sinh^2 v)\,[(1-n^2\sinh^2(v-i\theta))^{\frac{1}{2}}-i\kappa n\cosh(v-i\theta)]} \qquad (33)$$

$$+\frac{\kappa\gamma_2 r\sin\theta}{\pi^2}\int_1^\infty\frac{dp(p^2-1)^{\frac{1}{2}}(p^2+n^2)^{\frac{1}{2}}}{[\kappa^2(p^2+n^2)+1-p^2]\,\{(\gamma_2\tau-rp\cos\theta)^2+r^2(p^2+n^2)\sin^2\theta\}}\;.$$

It will again be noted that only the explicit residue terms are present on the z-axis away from the point source, all the integral terms vanish when $r = 0$, except at the source itself. This point is relevant when we consider the strength of the source in this two-medium problem.

Section 6: The flux associated with a point source moving on a plane interface

We return to the indirect method for calculating the source strength outlined in Section 2. Given the potential ϕ_1 in the range $0 < \theta < \pi/2$ we define the quantity

$$S_{Q_1} = -2\int_0^{\pi/2} \lim_{r\to 0} [\tau\phi_1]\, d\theta \tag{34}$$

which represents the flux per unit length produced in medium 1 by a moving line singularity whose potential Q_1 is given by the equation

$$Q_1 = -\frac{\kappa}{2\pi^2} \; \mathrm{Rl}\int_{-\infty}^{\infty} \frac{\ln\{\gamma_2\tau - r[p\cos\theta + \sin\theta(m^2p^2)^{\frac{1}{2}}]\}}{(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}} \;.$$

No matter what the value of V, we may easily calculate the value of S_{Q_1} as a function of τ from the equations (29), (31), (33), and (34) remembering that only the explicit residue terms make a contribution in the limit $r\to 0$. In each case there is the same discontinuity in the value of S_{Q_1} across the origin $\tau = 0$, so that the total volume flux from the point source into medium 1 is $\kappa/(1+\kappa)$.

Similarly we have the quantity

$$S_{Q_2} = -2\int_{-\pi/2}^{0} \lim_{r\to 0} [\tau\phi_2]\, d\theta \tag{35}$$

which represents the flux per unit length produced in medium 2 by the line source of potential Q_2 given by

$$Q_2 = -\frac{1}{2\pi^2} \text{Rl} \int_{-\infty}^{\infty} \frac{\ln\{\gamma_2\tau - r[p\cos\theta - \sin\theta(1-p^2)^{\frac{1}{2}}]\}\, dp}{(1-p^2)^{\frac{1}{2}} + \kappa(m^2-p^2)^{\frac{1}{2}}}$$

We find from the expressions (22), (25), (27) and (35) that the volume flux created in medium 2 by the moving point source is $1/(1+k)$.

The total volume flux produced by the point source is unity, it is apportioned in a definite ratio between the two media such that the rate at which mass is created by the source is the same for each medium regardless of the (constant) velocity of the source.

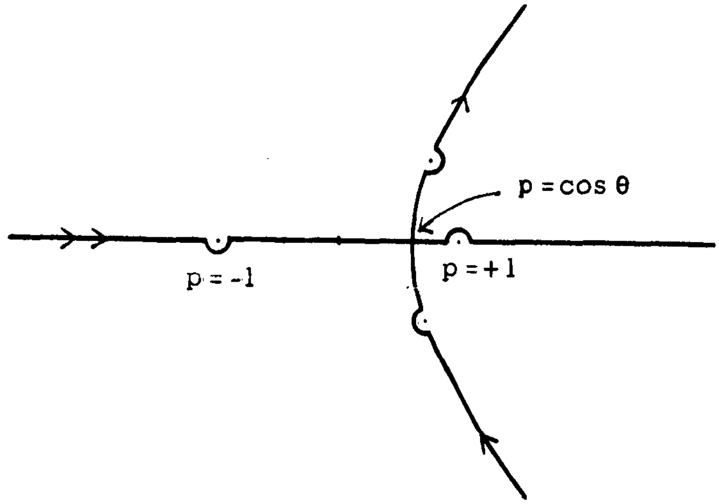

Figure 1

The original and subsequent integration paths for equation (7) in the supersonic case ($\tau > 0$) .

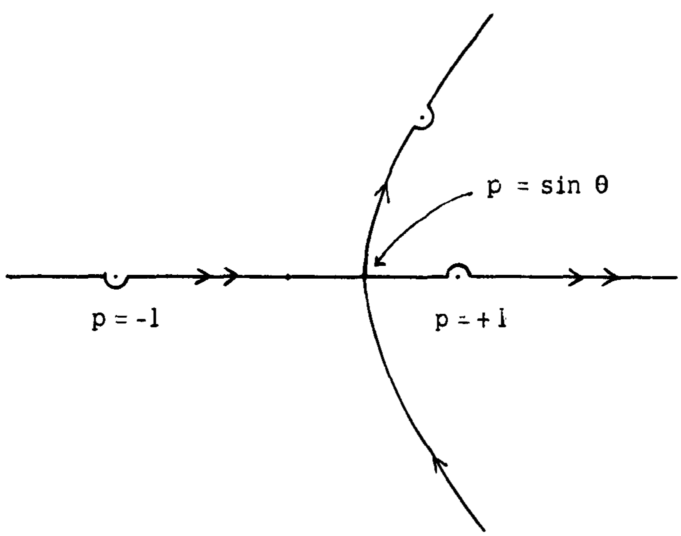

Figure 2a

The original and subsequent integration paths for equation (7) in the subsonic case ($\tau > 0$) .

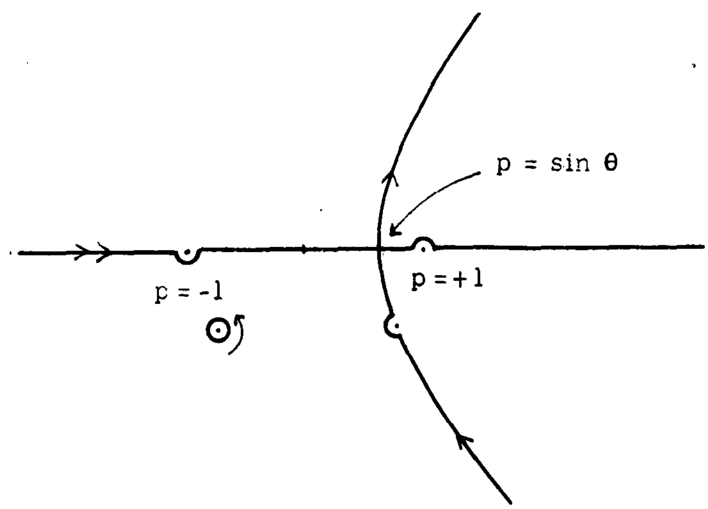

Figure 2b

The original and subsequent integration paths for equation (7) in the subsonic case ($\tau < 0$) .

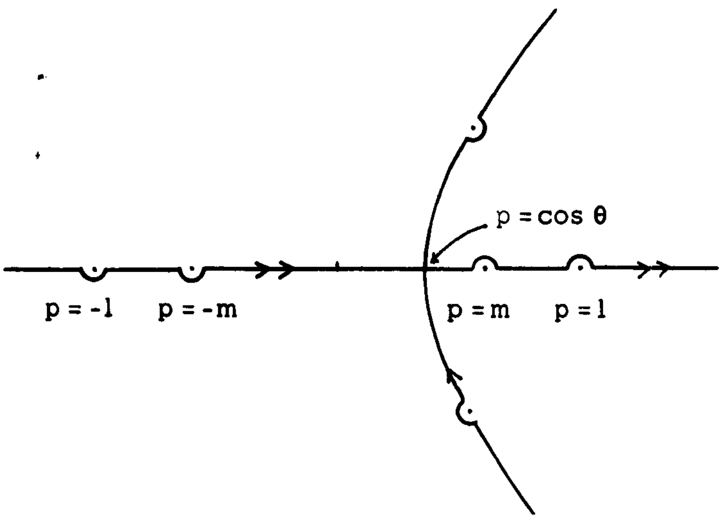

Figure 3a

The original and subsequent integration paths for equation (21) in the fully supersonic case ($\tau > 0$, $\cos\theta < m$) .

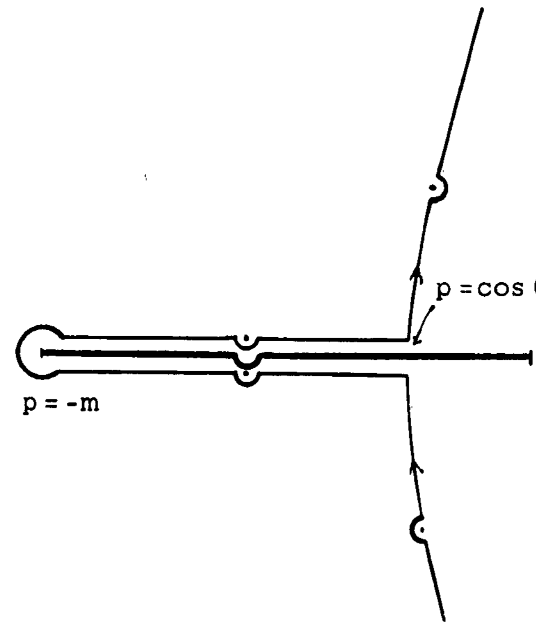

Figure 3b

The subsequent integration path for equation (21) in the fully supersonic case ($\tau > 0$, $\cos\theta > m$) .

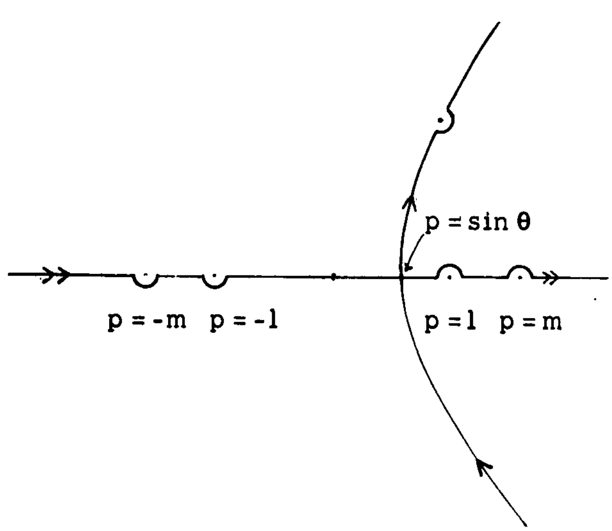

Figure 4a

The original and subsequent integration paths for equation (24) in the fully subsonic case ($\tau > 0$) .

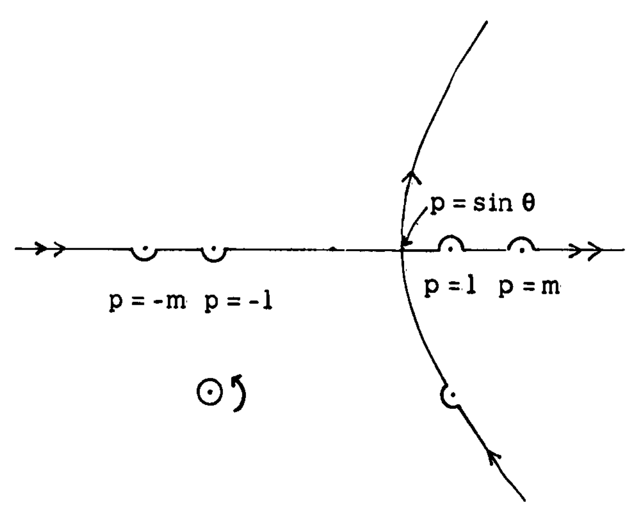

Figure 4b

The original and subsequent integration paths for equation (24) in the fully subsonic case ($\tau < 0$) .

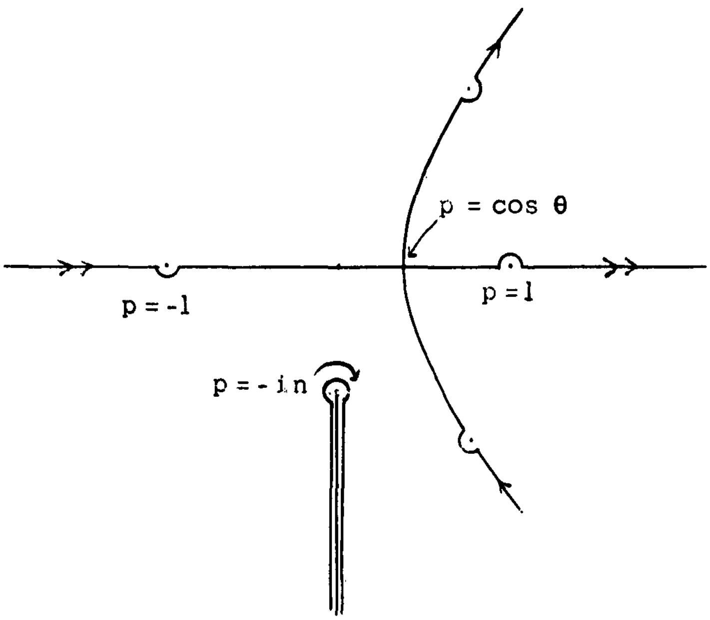

Figure 5a

The original and subsequent integration paths for equation (26) in the case of intermediate velocity ($\tau > 0$) .

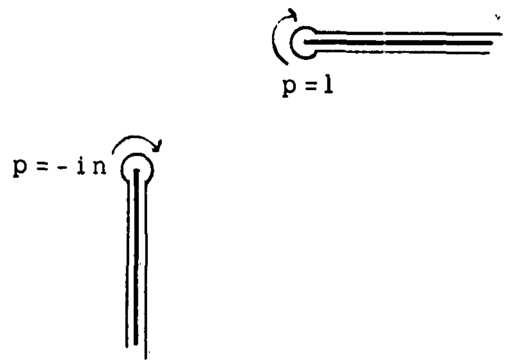

Figure 5b

The subsequent integration path for equation (26) in the case of intermediate velocity ($\tau < 0$) .

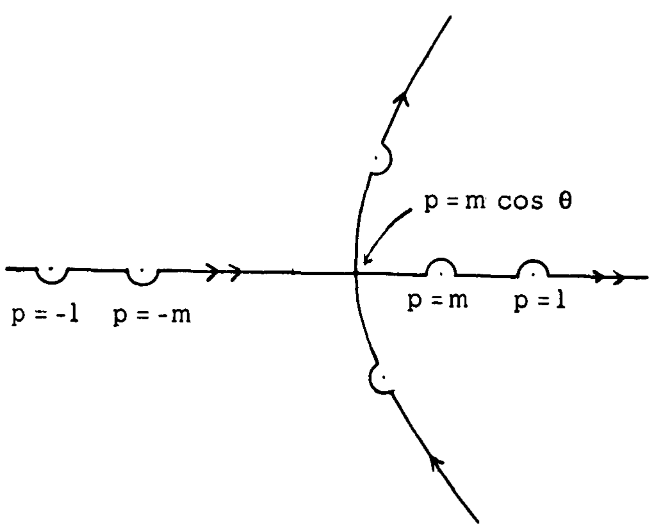

Figure 6

The initial and subsequent integration paths for equation (28) in the fully supersonic case ($\tau > 0$) .

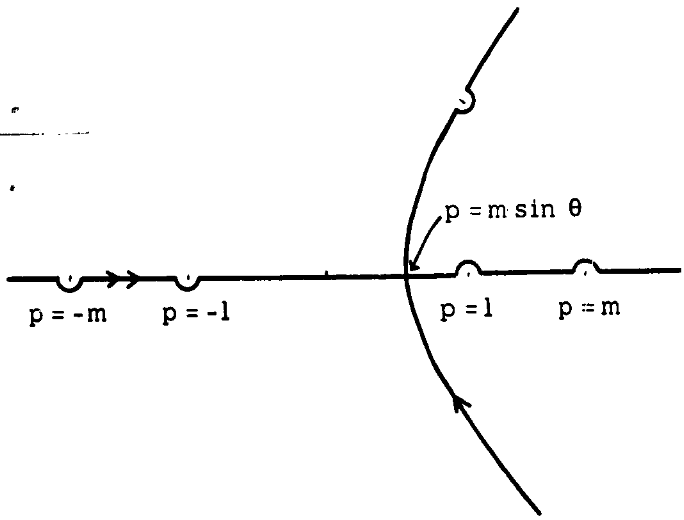

Figure 7a

The initial and subsequent integration paths for equation (30) in the fully subsonic case ($\tau > 0$, $m \sin \theta < 1$) .

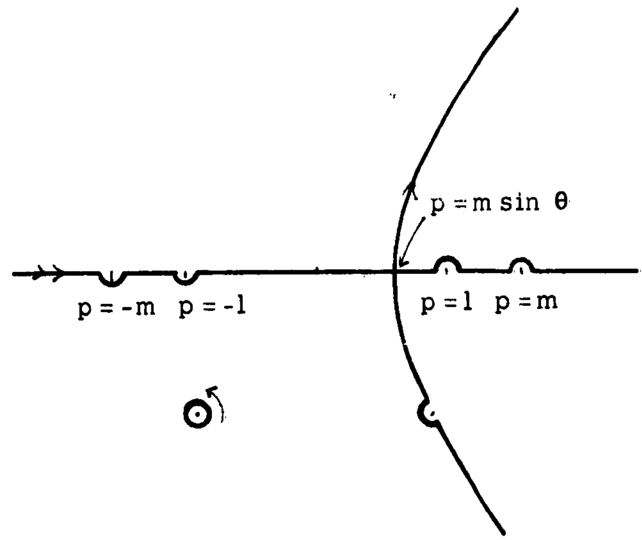

Figure 7b

The initial and subsequent integration paths for equation (30) in the fully subsonic case ($\tau < 0$, $m \sin \theta < 1$) .

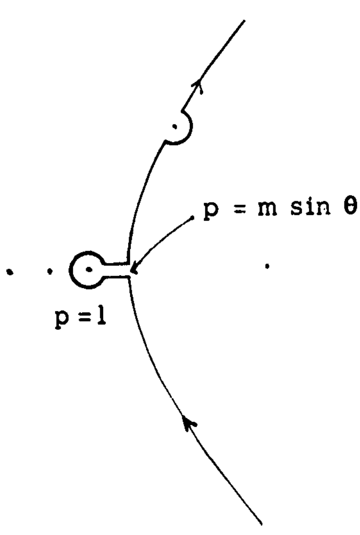

Figure 7c

The subsequent integration path for equation (30) in the fully subsonic case ($\tau > 0$, $m \sin \theta > 1$) .

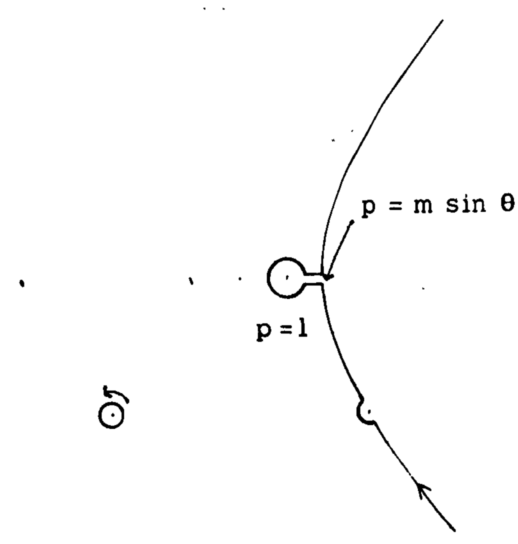

Figure 7d

The subsequent integration path for equation (30) in the fully subsonic case ($\tau < 0$, $m \sin \theta > 1$) .

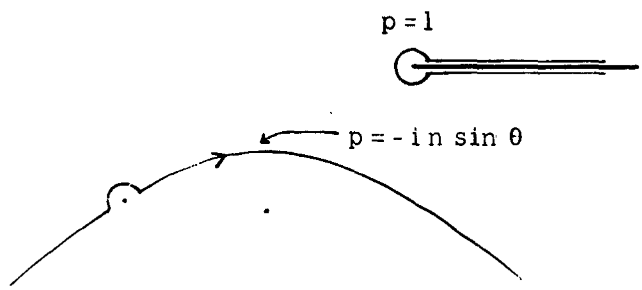

Figure 8

The subsequent integration path for equation (32) in the case of intermediate velocity.

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